PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 3, November 1988

ON COMPLEMENTED COPIES OF c_0 IN $L^p_X, 1 \leq p < \infty$

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(Communicated by William J. Davis)

Let (S, Σ, μ) be a not purely atomic measure space and X be a Banach space. In this note we want to show that if X contains a copy of c_0 then the usual Banach space of the Lebesgue-Bochner integrable functions L^p_X, $1 \leq p < \infty$, contains a complemented copy of c_0. Our result is similar in spirit to one obtained in [**1**] by Cembranos concerning the Banach space $C_X(K)$; in passing we observe that the Cembranos result has been extended in [**3**] to the case of ε-tensor products and then in [**2**] to the case of the Banach space of compact weak*-weak continuous operators.

In order to prove our theorem we need the definition of limited sets. A (bounded) subset M of a Banach space X is said to be limited if for each weak* null sequence $(x^*_n) \subset X^*$ we have $\lim_n \sup_{x \in M} |x^*_n(x)| = 0$. Further we use the following result obtained in [**2**].

LEMMA. *If X contains an unlimited sequence (x_n) that is equivalent to the unit basis of c_0, then X contains a complemented copy of c_0.*

Now, we are ready to show our theorem.

THEOREM. *Assume X contains a copy of c_0. Then L^p_X, $1 \leq p < \infty$, contains a complemented copy of c_0.*

PROOF. We shall construct a sequence of functions in L^p_X which is equivalent to the unit basis of c_0 and is not limited in L^p_X, so by virtue of the Lemma we will be done. Let (x_n) be a sequence in X equivalent to the unit basis of c_0 and (x^*_n) be a bounded sequence in X^* such that $x^*_m(x_n) = \delta_{mn}$. It suffices to consider the case of $[0,1]$ with Lebesgue measure. We consider Rademacher functions r_n and define a sequence (f_n) in L^p_X by putting $f_n = r_n x_n$ and a sequence in $(L^p_X)^*$ by putting $f^*_n = r_n x^*_n$. First of all we show that (f_n) is a sequence equivalent to the unit basis of c_0. Since (x_n) is a copy of the unit basis of c_0, there are $h_1, h_2 \in \mathbf{R}^+$ such that, for all finite sequences $(a_i)_{i=1}^s$ of real numbers, we have

$$h_1 \max_{1 \leq i \leq s} |a_i| \leq \left\| \sum_{i=1}^s a_i x_i \right\|_X \leq h_2 \max_{1 \leq i \leq s} |a_i|.$$

Since $|r_i(t)| = 1$ on $[0,1]$ for all $i \in N$, we have

$$h_1 \max_{1 \leq i \leq s} |a_i| \leq \left\| \sum_{i=1}^s a_i r_i(t) x_i \right\|_X \leq h_2 \max_{1 \leq i \leq s} |a_i|, \qquad t \in [0,1].$$

Received by the editors February 14, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46B20, 46B25.

Key words and phrases. Lebesgue-Bochner function spaces, complemented copies of c_0, limited sets.

This easily gives that

$$h_1 \max_{1\le i\le s} |a_i| \le \left\| \sum_{i=1}^{s} a_i f_i \right\|_{L_X^p} \le h_2 \max_{1\le i\le s} |a_i|,$$

i.e. (f_n) is equivalent to the unit basis of c_0. Now, we observe that

$$f_n^*(f_n) = \int_{[0,1]} x_n^*(x_n) r_n^2(t)\, dm = 1 \quad \text{for all } n \in N.$$

So it remains only to prove that $f_n^* \xrightarrow{w^*} 0$. To this purpose take $h \in L_X^p$ and observe that $|f_n^*(h)| = |\int_{[0,1]} x_n^*(h(t)) r_n(t)\, dm| \le \|x_n^*\| \, \|\int_{[0,1]} h(t) r_n(t)\, dm\|$ for all $n \in N$. Since (x_n^*) is bounded and moreover $\lim_n \|\int_{[0,1]} h(t) r_n(t)\, dm\| = 0$, we get $\lim_n f_n^*(h) = 0$. Arbitrariness of h in L_X^p gives that $f_n^* \xrightarrow{w^*} \theta$. Our proof is complete.

The above Theorem has the following Corollary.

COROLLARY. *Let X contain a copy of c_0. Then L_X^p, $1 \le p < \infty$, is neither a Grothendieck space nor a dual space.*

Finally, we want to thank Joe Diestel for suggesting a simplification of our first proof.

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